

ON THE DUAL OF THE MOBILE CONE.

SUNG RAK CHOI

ABSTRACT. We prove that the mobile cone and the cone of curves birationally movable in codimension 1 are dual to each other in the $(K + B)$ -negative part for a klt pair (X, B) . This duality of the cones gives a partial answer to the problem posed by Sam Payne. We also prove the cone theorem and the contraction theorem for the expanded cone of curves birationally movable in codimension 1.

1. INTRODUCTION

Let X be a normal projective algebraic variety defined over an algebraically closed field k (of characteristic 0). It is well-known that due to Kleiman-Seshadri, the cone of nef divisors $\text{Nef}(X) \subseteq N^1(X)$ and the cone of curves (often called the Mori cone) $\overline{NE}(X) \subseteq N_1(X)$ are dual to each other. It is also well-known that due to Boucksom-Demailly-Paun-Peternell [6], the cone of pseudoeffective divisors $\overline{\text{Eff}}(X) \subseteq N^1(X)$ and the cone of movable curves $\overline{NM}(X) \subseteq N_1(X)$ are dual to each other:

$$\begin{array}{ccc} \overline{\text{Eff}}(X) & \supseteq & \text{Nef}(X) \\ \text{dual} \updownarrow & & \text{dual} \updownarrow \\ \overline{NM}(X) & \subseteq & \overline{NE}(X). \end{array}$$

The next most important cone in $N^1(X)$ is probably the *mobile cone* $\overline{\text{Mob}}(X)$, the closed convex cone spanned by all the numerical classes of mobile divisors. Mobile divisors are the divisors whose \mathbb{R} -base loci (see Section 3) are of codimension ≥ 2 . The mobile cone $\overline{\text{Mob}}(X)$ is a subcone of $\overline{\text{Eff}}(X)$ which contains the nef cone $\text{Nef}(X)$: $\text{Nef}(X) \subseteq \overline{\text{Mob}}(X) \subseteq \overline{\text{Eff}}(X)$. It is natural to ask what the dual of the mobile cone $\overline{\text{Mob}}(X)$ is. In this paper, we will find a partial answer to this question.

A naive candidate for the dual of the mobile cone $\overline{\text{Mob}}(X)$ is the closed convex cone $\overline{NM}^1(X) \subseteq N_1(X)$ spanned by the classes of curves movable in codimension 1 subvarieties. However, Payne in [18] showed that in general the cone $\overline{NM}^1(X)$ is strictly smaller than the dual $\overline{\text{Mob}}(X)^\vee$. He also showed that in the case of complete \mathbb{Q} -factorial toric varieties, we have to also allow the classes of curves movable in codimension 1 subvarieties on \mathbb{Q} -factorial *small modifications* of X in order to obtain the correct dual of $\overline{\text{Mob}}(X)$ ([18, Theorem 2]). Following his ideas, we will give a partial generalization of his result for \mathbb{Q} -factorial klt pairs (X, B) where X is not

necessarily toric, which is valid in the $(K + B)$ -negative part of the cone (Theorem 1.1). This also gives a partial answer to the problem posed in [18] for \mathbb{Q} -factorial Fano type varieties (Corollary 3.11).

Let $f : X \dashrightarrow X'$ be a small birational map between \mathbb{Q} -factorial normal projective varieties. Since it is known that $N^1(X)$ and $N^1(X')$ are isomorphic under f_* [16, 12-2-1], their dual spaces $N_1(X)$ and $N_1(X')$, respectively, are also isomorphic: $N_1(X) \cong N_1(X')$. Under this isomorphism, a class $\alpha = [C] \in N_1(X')$ defined by a $\text{mov}^1(\text{movable in codimension } 1)\text{-curve } C$ on X' can be pulled back to $N_1(X)$ and we can simply consider α as a class in $N_1(X)$. The mov^1 -curve C on X' is called a $\text{b-mov}^1(\text{birationally movable in codimension } 1)\text{-curve}$ of X . We define $\overline{\text{bNM}}^1(X)$ as the closed convex cone in $N_1(X)$ spanned by all the classes of b-mov^1 -curves of X . See Section 3 for details.

We have the following partial duality result.

Theorem 1.1. *Let (X, B) be a \mathbb{Q} -factorial klt pair. Then*

$$\overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{Mob}}(X)^\vee = \overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{bNM}}^1(X).$$

In other words, the dual cone $\overline{\text{Mob}}(X)^\vee$ coincides with $\overline{\text{bNM}}^1(X)$ at least in some portion of the $(K + B)$ -negative part. Inspired by the results in [1] and [15], we also prove the following cone theorem for $\overline{\text{bNM}}^1(X)$.

Theorem 1.2 (The Cone Theorem for $\overline{\text{bNM}}^1(X)$). *Let (X, B) be a \mathbb{Q} -factorial klt pair. Then there exists a countable set of b-mov^1 -curves $\{C_i\}_{i \in I}$ of X such that*

$$\overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{bNM}}^1(X) = \overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\sum_{i \in I} \mathbb{R}_{\geq 0} \cdot [C_i]}$$

and for any ample H and any $\varepsilon > 0$, there exists a finite subset $J \subseteq I$ such that

$$\overline{\text{NE}}(X)_{K+B+\varepsilon H \geq 0} + \overline{\text{bNM}}^1(X) = \overline{\text{NE}}(X)_{K+B+\varepsilon H \geq 0} + \sum_{j \in J} \mathbb{R}_{\geq 0} \cdot [C_j].$$

The rays $\{R_i = \mathbb{R}_{\geq 0}[C_i]\}_{i \in I}$ in the first equality can accumulate only at the hyperplanes supporting both $\overline{\text{NE}}(X)_{K+B \geq 0}$ and $\overline{\text{bNM}}^1(X)$.

Note that this is actually a structure theorem for the expanded cone $\overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{bNM}}^1(X)$ (see Figure 1 in Section 4). We also prove the following contraction theorem for $\overline{\text{bNM}}^1(X)$. We call an extremal ray R of $\overline{\text{bNM}}^1(X)$ a mov^1 -co-extremal ray for (X, B) if it is $(K + B)$ -negative and it is also an extremal ray for the expanded cone $\overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{bNM}}^1(X)$. See Section 4 for details.

Theorem 1.3 (Contraction Theorem for mov^1 -co-extremal rays). *Let (X, B) be a \mathbb{Q} -factorial klt pair. Let R be a mov^1 -co-extremal ray of $\overline{\text{bNM}}^1(X)$ for (X, B) . Then the following hold:*

- (1) *there exists a small birational map $\varphi : X \dashrightarrow X'$ and a contraction $\psi : X' \rightarrow Y$ which is either a divisorial contraction or a Mori fiber space such that the mov^1 -co-extremal ray R is spanned by a mov^1 -curve C on X' if and only if C is contracted by ψ , and*
- (2) *the composition map $\psi \circ \varphi$ is uniquely determined by R .*

This paper is organized as follows:

In section 2, we review the definitions and properties of the non-ample locus $\mathbf{B}_+(D)$ and non-nef locus $\mathbf{B}_-(D)$ of divisors D . We also recall some necessary results from the theory of log minimal model program (LMMP). In section 3, we study the structure of the mobile cone $\overline{\text{Mob}}(X)$. The proof of Theorem 1.1 is given in this section. In section 4, we prove Theorem 1.2 and Theorem 1.3.

I would like to thank S. Boucksom for pointing out an error in the preliminary version of this paper and Z. Ran for answering my question. I also would like to thank V. Shokurov for the encouragement.

2. PRELIMINARIES

Let X be a normal projective variety. For a \mathbb{Z} -divisor D on X , its base locus $\text{Bs}(D)$ is defined as the support of the intersection of the elements in the usual \mathbb{Z} -linear system $|D| = \{D' \in \text{Div}_{\mathbb{Z}}(X) \mid D \sim D' \geq 0\}$. For a \mathbb{Q} -divisor D , the *stable base locus* of D is defined as $\mathbf{B}(D) := \bigcap_m \text{Bs}(mD)$ where the intersection is taken over the positive integers m such that mD are integral. For an \mathbb{R} -divisor D on X , the \mathbb{R} -linear system is defined as $|D|_{\mathbb{R}} := \{D' \in \text{Div}_{\mathbb{R}}(X) \mid D \sim_{\mathbb{R}} D' \geq 0\}$ and its *\mathbb{R} -stable base locus* as $\mathbf{B}_{\mathbb{R}}(D) := (\cap |D|_{\mathbb{R}})_{\text{red}}$. Clearly, $\mathbf{B}_{\mathbb{R}}(D) \subseteq \mathbf{B}(D)$ for a \mathbb{Q} -divisor D . From now on, we always use \mathbb{R} -divisors unless otherwise stated.

For a divisor D on X , we define the *non-ample locus* (or *augmented base locus*) of D as

$$\mathbf{B}_+(D) := \bigcap_{\text{ample } A} \mathbf{B}(D - A)$$

where the intersection is taken over all ample divisors A such that $D - A$ are \mathbb{Q} -divisors. Note that $\mathbf{B}_+(D) = X$ if D is not big. As the name suggests, D is ample if and only if $\mathbf{B}_+(D) = \emptyset$. We define the *non-nef locus* (or *diminished base locus*) of D as

$$\mathbf{B}_-(D) := \bigcup_{\text{ample } A} \mathbf{B}(D + A)$$

where the union is taken over all ample divisors A such that $D + A$ are \mathbb{Q} -divisors. Note that $\mathbf{B}_-(D) = X$ if D is not pseudoeffective. It is easy to see that D is nef if and only if $\mathbf{B}_-(D) = \emptyset$. It is known that $\mathbf{B}_+(D) = \mathbf{B}_+(D - A) = \mathbf{B}_-(D - A)$ for any sufficiently small ample divisor A [8, Proposition 1.21]. It is also well known that the base loci $\mathbf{B}_+(D), \mathbf{B}_-(D)$ depend only on the numerical class of D whereas $\mathbf{B}(D)$ is

not in general. The non-ample locus $\mathbf{B}_+(D)$ is Zariski closed for any D whereas the non-nef locus $\mathbf{B}_-(D)$ is *a priori* not in general.

Remark 2.1. The following inclusions are easy to verify and often useful: for any ample divisor A ,

$$\mathbf{B}_+(D + A) \subseteq \mathbf{B}_-(D), \quad \mathbf{B}(D + A) \subseteq \mathbf{B}_-(D), \quad \text{and} \quad \mathbf{B}_+(D + A) \subseteq \mathbf{B}(D).$$

It is easy to see that if $V \subseteq \mathbf{B}_-(D)$ for some subvariety V of X , then there exists a small ample divisor A such that $V \subseteq \mathbf{B}_+(D + A)$. Thus we can also define the non-nef locus as $\mathbf{B}_-(D) := \cup \mathbf{B}_+(D + A)$ where the union is taken over all ample divisors A .

See [5],[7],[8],[14] for more details about the non-ample loci and non-nef loci.

A big divisor D is \mathbb{R} -mobile if $\mathbf{B}_{\mathbb{R}}(D)$ does not have a divisorial component. We define the cones in the numerical space $N^1(X)$:

$$\begin{aligned} \text{Amp}(X) &:= \{[D] \in N^1(X) \mid D \text{ is ample}\}, \\ \text{Mob}(X) &:= \{[D] \in N^1(X) \mid D \equiv D' \text{ for some } \mathbb{R}\text{-mobile } D'\}, \\ \text{Eff}(X) &:= \{[D] \in N^1(X) \mid D \equiv D' \text{ for some effective } D'\}. \end{aligned}$$

Their closures $\text{Nef}(X) = \overline{\text{Amp}}(X)$, $\overline{\text{Mob}}(X)$, and $\overline{\text{Eff}}(X)$ are called the *nef cone*, *mobile cone*, and *pseudoeffective cone*, respectively. They satisfy the inclusion: $\text{Nef}(X) \subseteq \overline{\text{Mob}}(X) \subseteq \overline{\text{Eff}}(X)$. We will study the mobile cone $\overline{\text{Mob}}(X)$ in detail using the base loci $\mathbf{B}_-, \mathbf{B}_+$ in Section 3.

For a cone $V \subseteq N_1(X)$, a divisor D and $\square \in \{=, <, >, \geq, \leq\}$, we define

$$V_{D \square 0} := V \cap \{C \in N_1(X) \mid D \cdot C \square 0\}.$$

An *extremal face* F of a closed convex cone V satisfies the two conditions 1) F is a convex subcone of V , and 2) if $v + u \in F$ for $u, v \in V$, then $u, v \in F$. A one dimensional extremal face is called an *extremal ray*.

We use the standard notions of singularities of pairs (X, B) in the log minimal model program (LMMP, for short) [12],[10]. We briefly recall the basics of the LMMP. For an exceptional prime divisor E over X , $a(E, X, B)$ denotes the log discrepancy of (X, B) at E .

Definition 2.2. Let (X, B) be a \mathbb{Q} -factorial klt pair and let $\varphi : X \dashrightarrow Y$ be a birational map to a projective \mathbb{Q} -factorial variety Y . Let $B_Y := f_*B$.

- (1) A pair (Y, B_Y) is called a *log terminal model* of (X, B) if the pair (Y, B_Y) is klt, $K_Y + B_Y$ is nef, and the inequality $1 - \text{mult}_E B < a(E, Y, B_Y)$ holds for any φ -exceptional prime divisor E .
- (2) A pair (Y, B_Y) equipped with a fibration $Y \rightarrow T$ is called a *Mori log fibration* of (X, B) if (Y, B_Y) is klt, $\dim T < \dim Y$, the relative Picard number $\rho(Y/T) = 1$, and $-(K_Y + B_Y)$ is ample over T .

- (3) A resulting model of (X, B) is either a log terminal model (1) or a Mori log fibration (2).

By the LMMP, any \mathbb{Q} -factorial klt pair (X, B) is expected to have a resulting model and it cannot have both resulting models simultaneously [20, 2.4.1]. It is known that (X, B) has a log terminal model as its resulting model if and only if $K + B$ is pseudoeffective [4].

We will use the following lemma often.

Lemma 2.3. [14, Example 9.2.29] *Let (X, B) be a klt pair and H be an ample divisor on X . Then there exists an effective divisor $H' \sim_{\mathbb{R}} H$ such that $(X, B + H')$ is klt.*

If the pairs $(X, B), (X, B')$ are klt and $B \sim_{\mathbb{R}} B'$, then (X, B) and (X, B') have the same resulting models by the LMMP. Therefore by Lemma 2.3, given a klt pair (X, B) and an ample divisor H , we may assume that $(X, B + H)$ is klt in order to run the LMMP or to study the resulting models of $(X, B + H)$.

We review some necessary results from [4]. First, we state an important result about the decomposition of the following set:

$$\mathcal{E}_H := \{B \in U \mid B \geq 0, (X, B) \text{ is klt and } K + B + H \text{ is pseudo-effective} \}$$

where U is a finite dimensional subspace of real Weil divisors which is defined over the rationals and H is an ample divisor on X .

Theorem 2.4. [4, Corollary 1.1.5] *Let H be a rational ample divisor and suppose that for some $B_0 \in \mathcal{E}_H$, the pair (X, B_0) is klt. Then for any $B \in \mathcal{E}_H$, the pair (X, B) has a log terminal model. Furthermore, there exist finitely many birational maps $\varphi_i : X \dashrightarrow X_i$ ($1 \leq i \leq p$) and the set \mathcal{E}_H is decomposed into finitely many rational polytopes \mathcal{W}_i*

$$\mathcal{E}_H = \bigcup_i^p \mathcal{W}_i,$$

satisfying the following condition: if, for $B \in \mathcal{E}_H$, there exists a birational contraction $\varphi : X \dashrightarrow Y$ which is a log terminal model of (X, B) , then $\varphi = \varphi_i$ for some $1 \leq i \leq p$.

In [21], the similar decomposition problem (which is called the *geography*) is also studied in detail in terms of b-divisors. The following is also the consequence of [4].

Theorem 2.5. [4] *Let X be a projective variety and let $D := K + B$ where (X, B) is klt, B is big and $K + B$ is pseudoeffective. Then there exists a birational map $\Phi : X \dashrightarrow X'$ such that*

- (1) $D' := \Phi_* D$ is nef,
- (2) $D \geq D'$ (i.e. the inequality is satisfied after pulling back to the graph),
- (3) Φ is surjective in codimension 1, and

- (4) a prime divisor E of X is contracted by Φ if and only if it is a divisorial component of $\mathbf{B}_-(D)$.

Furthermore, if $K + B$ is big, then there exists a contraction $\Psi : X' \rightarrow Y$, (which is the unique log canonical model of (X, B)) where D' vanishes on the curves contracted by Ψ .

Condition 4 is a well-known reformulation of the “strict negativity” condition of [4] by Kawamata [11, Lemma 2]. The prime divisors supported in the *numerically fixed part* of D in [11] coincide with the divisorial components of $\mathbf{B}_-(D)$. By conditions 3 and 4, the map Φ is an isomorphism in codimension 1 if $D = K + B \in \overline{\text{Mob}}(X)$.

3. THE MOBILE CONE $\overline{\text{Mob}}(X)$

We will characterize the mobile cone $\overline{\text{Mob}}(X)$ using the non-ample locus \mathbf{B}_+ and non-nef locus \mathbf{B}_- . We will also study the dual of $\overline{\text{Mob}}(X)$.

Let $\text{Mob}_+(X)$ (resp. $\text{Mob}_-(X)$) be the cone in $N^1(X)$ spanned by the divisors D such that $\mathbf{B}_+(D)$ (resp. $\mathbf{B}_-(D)$) do not contain codimension 1 subvarieties. It is easy to see that $\mathbf{B}_-(D) \subseteq \mathbf{B}_{\mathbb{R}}(D) \subseteq \mathbf{B}_+(D)$. Therefore $\text{Mob}_+(X) \subseteq \text{Mob}(X) \subseteq \text{Mob}_-(X)$.

Lemma 3.1. *Let $\text{Mob}_-(X), \text{Mob}_+(X)$ be the cones in $N^1(X)$ defined above.*

- (1) *Let $D \in \overline{\text{Mob}}_+(X)$. Then $D + A \in \text{Mob}_+(X)$ for any ample divisor A .*
- (2) *The cone $\text{Mob}_+(X)$ is open and $\overline{\text{Mob}}_+(X) = \text{Mob}_-(X)$. In particular, the cone $\text{Mob}_-(X)$ is closed and*

$$\overline{\text{Mob}}_+(X) = \overline{\text{Mob}}(X) = \text{Mob}_-(X).$$

Furthermore, $\text{Int } \text{Mob}_-(X) = \text{Mob}_+(X)$.

Proof. (1) Since $D \in \overline{\text{Mob}}_+(X)$, there exists a sequence $D_i \in \text{Mob}_+(X)$ such that $D_i \rightarrow D$ as $i \rightarrow \infty$. For a fixed ample divisor A , by taking i sufficiently large, we may assume that $A - (D_i - D)$ is ample for all i . Thus $\mathbf{B}_+(D_i) \supseteq \mathbf{B}_+(D_i + A - (D_i - D)) = \mathbf{B}_+(D + A)$ and $D + A \in \text{Mob}_+(X)$.

(2) The openness of the cone $\text{Mob}_+(X)$ follows from [8, Corollary 1.6]: there exists a small open neighborhood N of D such that for any $D' \in N$, $\mathbf{B}_+(D') \subseteq \mathbf{B}_+(D)$.

To prove the inclusion $\overline{\text{Mob}}_+(X) \subseteq \text{Mob}_-(X)$, let $D \in \overline{\text{Mob}}_+(X)$. If $D \notin \text{Mob}_-(X)$, then there exists a codimension 1 subvariety E of X such that $E \subseteq \mathbf{B}_-(D)$. By remark 2.1, $E \subseteq \mathbf{B}_+(D + A)$ for some ample divisor A , but it is a contradiction to (1). The inclusion $\overline{\text{Mob}}_+(X) \supseteq \text{Mob}_-(X)$ can be seen as follows. Let $D \in \text{Mob}_-(X)$. Then for a fixed ample divisor A , $\{D + \frac{1}{m}A\}$ is a sequence in $\text{Mob}_+(X)$ since $\mathbf{B}_+(D + \frac{1}{m}A) \subseteq \mathbf{B}_-(D)$ (Remark 2.1). Thus the limit of the sequence must belong to $\overline{\text{Mob}}_+(X)$, i.e., $D \in \overline{\text{Mob}}_+(X)$. \square

Lemma 3.2. *Let D be a big divisor such that $D \in \partial \overline{\text{Mob}}(X)$. Then there exists a divisorial component $E \subseteq \mathbf{B}_+(D)$ such that $E \not\subseteq \mathbf{B}_-(D)$. In particular, for any ample divisor A on X , $D + A$ is \mathbb{R} -mobile and $D - A$ is not \mathbb{R} -mobile.*

Proof. If $D \in \overline{\partial \text{Mob}}(X)$, then by Lemma 3.1 $D \notin \text{Mob}_+(X)$ and $D \in \text{Mob}_-(X)$. Thus there exists a subvariety E of codimension 1 such that $E \subseteq \mathbf{B}_+(D)$ and $E \not\subseteq \mathbf{B}_-(D)$. Since D is big, E is an irreducible component of $\mathbf{B}_+(D)$. It is easy to see that $D + A$ is \mathbb{R} -mobile by the definition of $\mathbf{B}_-(D)$ and that $D - A$ is not \mathbb{R} -mobile since $\mathbf{B}_+(D) \subseteq \mathbf{B}_{\mathbb{R}}(D - A)$. \square

Proposition 3.3. *Let $f : X \dashrightarrow X'$ be a small birational map between projective \mathbb{Q} -factorial varieties. Suppose that for a big divisor D on X , there exists a divisorial component $E \subseteq \mathbf{B}_+(D)$. Then $E' := f_*E$ is also a divisorial component of $\mathbf{B}_+(D')$ where $D' = f_*D$.*

Proof. Let W be a common resolution of X and X' with $p : W \rightarrow X$ and $q : W \rightarrow X'$. By Proposition 1.5 of [5], we have

$$\begin{aligned} \mathbf{B}_+(p^*(D)) &= p^{-1}(\mathbf{B}_+(D)) \cup \text{Exc}(p), \\ \mathbf{B}_+(q^*(D')) &= q^{-1}(\mathbf{B}_+(D')) \cup \text{Exc}(q), \end{aligned}$$

and $\mathbf{B}_+(p^*(D)) = \mathbf{B}_+(q^*(D'))$. If $E \subseteq \mathbf{B}_+(D)$ is a divisorial component, then $E_W := p_*^{-1}E$ is a divisorial component of $\mathbf{B}_+(p^*(D))$. The divisor E_W is not q -exceptional because X is isomorphic to X' in codimension 1. Thus $E' = q_*(E_W)$ is also a divisorial component of $\mathbf{B}_+(D')$. \square

Nakamaye gave another characterization of the non-ample locus $\mathbf{B}_+(D)$ when D is nef. We define the *null locus* $\text{Null}(D)$ of a nef and big divisor D as $\text{Null}(D) := \bigcup_V \{V \subseteq X \mid D^k \cdot V = 0 \text{ where } \dim V = k > 0\}$.

Theorem 3.4 (Nakamaye's theorem). [14, Theorem 10.3.5], [17] *Let D be a nef and big divisor on X . Then*

$$\mathbf{B}_+(D) = \text{Null}(D).$$

This implies $D^{\dim V} \cdot V = 0$ for any irreducible component V of $\mathbf{B}_+(D)$. We will also need the following result due to Khovanskii and Teissier.

Theorem 3.5 (Khovanskii-Teissier inequality). [13, Theorem 1.6.1] *Let X be a variety of dimension d and D_i be nef divisors. Then*

$$D_1 \cdot D_2 \cdots D_d \geq (D_1^d)^{\frac{1}{d}} \cdot (D_2^d)^{\frac{1}{d}} \cdots (D_d^d)^{\frac{1}{d}}.$$

Taking into consideration of Payne's idea [18], we define the following curves.

Definition 3.6. *Let X be a \mathbb{Q} -factorial normal variety of dimension d .*

- *A curve C on X is called a movable curve if it is a member of a family of curves covering X .*
- *A curve C on X is called a mov^1 (movable in codimension 1)-curve if it is a member of a family of curves covering a subvariety of codimension 1.*

- A mov^1 -curve C on some \mathbb{Q} -factorial X' which is isomorphic to X in codimension 1 is called a b-mov^1 (birationally movable in codimension 1)-curve of X .

Note that as explained in Introduction, a b-mov^1 -curve C of X defines a class $\alpha = [C] \in N_1(X)$ even though the curve C may not be defined on X . Thus we may treat a b-mov^1 -curve C as a class in $N_1(X)$. We let $\text{NM}(X)$, $\text{NM}^1(X)$ be the cones in $N_1(X)$ that are spanned by the classes of movable curves and mov^1 -curves on X , respectively. We define $\text{NM}^1(X, X')$ as the image in $N_1(X)$ of the cone $\text{NM}(X')$ under the isomorphism $N_1(X') \cong N_1(X)$. Lastly, we define $\text{bNM}^1(X)$ as the cone in $N_1(X)$ spanned by b-mov^1 -curves of X . It is easy to see that

$$\overline{\text{bNM}}^1(X) = \overline{\sum_{X \dashrightarrow X'} \text{NM}^1(X, X')},$$

where the summation is taken over all \mathbb{Q} -factorial X' that are isomorphic to X in codimension 1. By definition, a movable curve is mov^1 and a mov^1 -curve is a b-mov^1 -curve of X . Thus

$$\overline{\text{NM}}(X) \subseteq \overline{\text{NM}}^1(X) \subseteq \overline{\text{bNM}}^1(X).$$

It is important to note that the inclusion on the right is strict in general by Payne's counterexample [18, Example 1].

Theorem 3.7. *The following hold:*

- (1) $\text{Nef}(X) = \overline{\text{NE}}(X)^\vee$.
- (2) $\overline{\text{Eff}}(X) = \overline{\text{NM}}(X)^\vee$.

Proof. (1) It is a well known result due to Kleiman-Seshadri. See [13, Proposition 1.4.28]. (2) It is the main result of [6] for smooth varieties. The result also holds for \mathbb{Q} -factorial varieties. \square

According to Theorem 1.1, the cones $\overline{\text{Mob}}(X)$ and $\overline{\text{bNM}}^1(X)$ are dual to each other at least in some part of the cones. In order to prove Theorem 1.1, we prove the following equivalent dual statement:

$$(*) \quad \begin{array}{l} \text{the cones } \overline{\text{Mob}}(X) \text{ and } \overline{\text{bNM}}^1(X)^\vee \text{ coincide inside the convex cone} \\ P = \text{Nef}(X) + \mathbb{R}_{\geq 0} \cdot [K + B]. \end{array}$$

We start with an easy observation.

Lemma 3.8. *We have the following nonnegative intersection pairing:*

$$(\alpha, \beta) \in \overline{\text{Mob}}(X) \times \overline{\text{bNM}}^1(X) \longmapsto \alpha \cdot \beta \geq 0.$$

Proof. Let D be an \mathbb{R} -mobile divisor and C be a b-mov^1 -curve on X . Since the numerical classes in $N_1(X)$ are preserved under a small birational map, we may assume that C is a mov^1 -curve on X . Then since C moves in a family of curves

covering a subvariety of codimension 1, we may assume that C is disjoint from the base locus of D which is of codimension ≥ 2 . Thus $C \cdot D \geq 0$. The classes α and β are the limits of the classes of such curve C and divisor D . Therefore $\alpha \cdot \beta \geq 0$ by continuity. \square

Proof of Theorem 1.1. (Step 1) As we stated above, we prove the dual statement (*). By Lemma 3.8, we have $\overline{\text{Mob}}(X) \subseteq \overline{\text{bNM}}^1(X)^\vee$. This in particular implies

$$\overline{\text{Mob}}(X) \cap P \subseteq \overline{\text{bNM}}^1(X)^\vee \cap P,$$

where $P = \overline{\text{Nef}}(X) + \mathbb{R}_{\geq 0} \cdot [K + B]$. Suppose that the strict inclusion \subsetneq holds. Note that since $\overline{\text{bNM}}^1(X) \supseteq \overline{\text{NM}}(X)$, we have $\overline{\text{bNM}}^1(X)^\vee \subseteq \overline{\text{Eff}}(X)$ by (2) of Theorem 3.7. Note also that $\overline{\text{bNM}}^1(X)^\vee = \bigcap \overline{\text{NM}}^1(X, X')^\vee$, where the intersection is taken over all \mathbb{Q} -factorial X' that are isomorphic to X in codimension 1. Therefore, if the inclusion above is strict, then there exists a big divisor $D \in \partial \overline{\text{Mob}}(X) \cap \text{Int } P$ and $D \in \text{Int}(\overline{\text{NM}}^1(X, X')^\vee)$ for any \mathbb{Q} -factorial X' which is isomorphic to X in codimension 1.

(Step 2) There exists an ample divisor H such that $rD \equiv K + B + H$ for some $r > 0$. By rescaling, we may assume that $D \equiv K + B + H$. By Lemma 2.3, we may assume that $(X, B + H)$ is klt. By Theorem 2.5, there exists a log terminal model $f : X \dashrightarrow Y$ of $(X, B + H)$ which is an isomorphism in codimension 1.

(Step 3) Since $D \in \partial \overline{\text{Mob}}(X)$ and D is big, there exists a divisorial component $E \subseteq \mathbf{B}_+(D)$ (Lemma 3.2) and since the modification f is small, $E_Y := f_*E$ is also a divisorial component of $\mathbf{B}_+(D_Y)$ (Proposition 3.3). This implies that $D_Y \notin \text{Int } \overline{\text{Mob}}(Y)$. However, from Step 1, we have $D_Y \in \text{Int}(\overline{\text{NM}}^1(Y)^\vee)$. Since D_Y is also nef, by Lemma 3.9 we must have $D_Y \in \text{Int } \overline{\text{Mob}}(Y)$, and this is a contradiction. \square

The following lemma will finish the above proof.

Lemma 3.9. *If X is a projective \mathbb{Q} -factorial variety of dimension n , then we have*

$$\text{Nef}(X) \cap \text{Int}(\overline{\text{NM}}^1(X)^\vee) \subseteq \text{Int } \overline{\text{Mob}}(X).$$

Proof. Let $D \in \text{Nef}(X) \cap \text{Int}(\overline{\text{NM}}^1(X)^\vee)$. Note that D is big by [6] ((2) of Theorem 3.7). If D does not belong to $\text{Int } \overline{\text{Mob}}(X)$, then there exists a divisorial component $E \subseteq \mathbf{B}_+(D)$ and by Nakamaye's theorem (Theorem 3.4), $D^{n-1} \cdot E = 0$. Since $D \in \text{Int}(\overline{\text{NM}}^1(X)^\vee)$, there exists some ample \mathbb{Q} -divisor A such that $(D - A) \cdot C \geq 0$ for all mov^1 -curves C on X . If we apply this to the mov^1 -curve $C := (D + \lambda A)^{n-2} \cdot E$, then we obtain

$$(D + \lambda A)^{n-1} \cdot E \geq A \cdot (D + \lambda A)^{n-2} \cdot E.$$

Hence by the Khovanskii-Teissier inequality (Theorem 3.5),

$$(D + \lambda A)^{n-1} \cdot E \geq (A^{n-1} \cdot E)^{\frac{1}{n-1}} \cdot ((D + \lambda A)^{n-1} \cdot E)^{\frac{n-2}{n-1}},$$

and $(D + \lambda A)^{n-1} \cdot E \geq A^{n-1} \cdot E$. This shows that by taking the limit $\lambda \rightarrow 0$, we get a contradiction

$$0 \leftarrow (D + \lambda A)^{n-1} \cdot E \geq A^{n-1} \cdot E > 0.$$

□

We give a partial affirmative answer to the problem posed in [18] for the following type of varieties.

Definition 3.10. A \mathbb{Q} -factorial variety X is said to be of Fano type (FT) if there exists a boundary \mathbb{Q} -divisor B on X such that (X, B) is klt, $K + B \equiv 0$ and the divisors in the support of B generate $N^1(X)$.

See [19, Lemma-Definition 2.8] for equivalent definitions.

Corollary 3.11. For a \mathbb{Q} -factorial FT variety X , the following duality holds:

$$\overline{\text{Mob}}(X)^\vee = \overline{\text{bNM}}^1(X).$$

Furthermore, the cones $\overline{\text{Mob}}(X)$ and $\overline{\text{bNM}}^1(X)$ are closed convex and rational polyhedral.

Proof. There exists an effective boundary \mathbb{Q} -divisor B such that (X, B) is klt, $K + B \equiv 0$ and the components of B generate $N^1(X)$. There exists an effective ample divisor A such that $\text{Supp } A = \text{Supp } B$. The pair $(X, B - \varepsilon A)$ is klt for sufficiently small $\varepsilon > 0$ and $-(K + B - \varepsilon A)$ is ample. Therefore, the cone $\overline{\text{NE}}(X)$ is $(K + B - \varepsilon A)$ -negative and the equality follows immediately from Theorem 1.1 and Theorem 1.2. The last statement follows from the rational polyhedral property and the finiteness of the decomposition in Theorem 2.4. See also [21, Corollary 4.5]. □

Remark 3.12. In [18], it is shown that for complete \mathbb{Q} -factorial toric varieties X and $0 \leq k \leq \dim X$, the duality holds between the closed cone in $N^1(X)$ spanned by divisors that are ample in codimension k and the closed cone in $N_1(X)$ spanned by the curves that are birationally movable in codimension k . (see [18, Theorem 2]). Payne asks if this is also true for general non-toric varieties. Note that the two extreme cases $k = 0$ and $k = \dim X$ are true by Theorem 3.7. Corollary 3.11 gives an affirmative answer to this problem for \mathbb{Q} -factorial FT varieties X for the case $k = \dim X - 1$. It is also easy to see from the proof of Theorem 1.1 that the same duality holds for Mori dream spaces [9]. Indeed, the duality holds in the part of the cone $\overline{\text{Mob}}(X)$ where we can run the MMP. In [18], it is also explained that considering only the mov^1 -curves in Theorem 1.1 is not enough in general (see [18, Example 1]).

4. CONE THEOREMS

Inspired by the results in [1] and [15], we prove the cone theorem (Theorem 1.2) and the contraction theorem (Theorem 1.3) for the cone $\overline{\text{bNM}}^1(X)$ in this section.

Let (X, B) be a \mathbb{Q} -factorial klt pair. In the space $N_1(X)$, we consider the following two convex cones: (see Figure 1)

$$\begin{aligned} V(X, B)(= V) &:= \overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{bNM}}^1(X), \\ V'(X, B)(= V') &:= \overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{NM}}(X). \end{aligned}$$

An extremal face F of $\overline{\text{bNM}}^1(X)$ is called a *mov¹-co-extremal face* for the pair (X, B) if F is a $(K + B)$ -negative extremal face of V . A divisor D which is positive on $\overline{\text{NE}}(X)_{K+B \geq 0} \setminus \{0\}$ and such that the plane $\{\alpha \in N_1(X) | \alpha \cdot D = 0\}$ supports the cone V exactly at a *mov¹-co-extremal face* F is called a *mov¹-co-bounding divisor* of F . An extremal face F' of $\overline{\text{NM}}(X)$ is called a *co-extremal face* for the pair (X, B) if F' is a $(K + B)$ -negative extremal face of V' [3]. A divisor D which is positive on $\overline{\text{NE}}(X)_{K+B \geq 0} \setminus \{0\}$ and such that the plane $\{\alpha \in N_1(X) | \alpha \cdot D = 0\}$ supports the cone V' exactly at a *co-extremal face* F' is called a *co-bounding divisor* of F' .

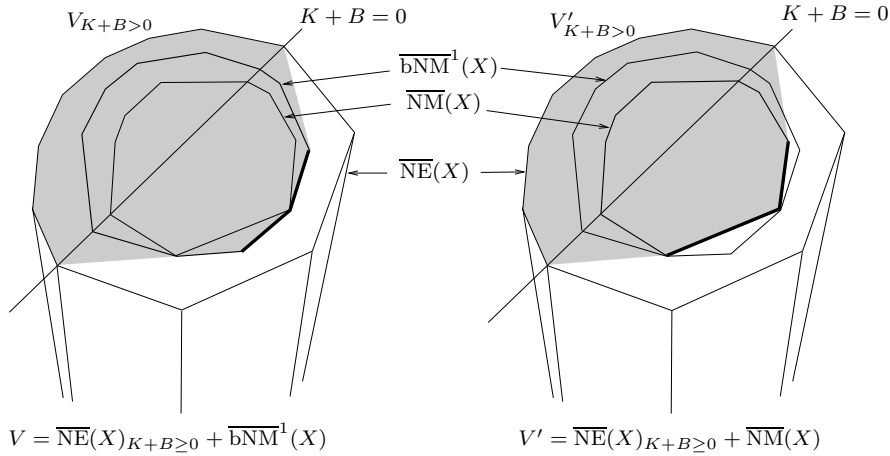


FIGURE 1

As illustrated in Figure 1, an extremal face of $\overline{\text{bNM}}^1(X)$ (resp. $\overline{\text{NM}}(X)$) in $\overline{\text{NE}}_{K+B < 0}(X)$ is not necessarily a *mov¹-co-extremal face* (resp. a *co-extremal face*). Note also that a *co-extremal face* of $\overline{\text{NM}}(X)$ can coincide with a *mov¹-co-extremal face* of $\overline{\text{bNM}}^1(X)$. It is also easy to see that the cone $\overline{\text{bNM}}^1(X)$ has a *mov¹-co-extremal ray* for (X, B) if and only if $K + B \notin \overline{\text{Mob}}(X)$ by Theorem 1.1 and the cone $\overline{\text{NM}}(X)$ has a *co-extremal ray* for (X, B) if and only if $K + B \notin \overline{\text{Eff}}(X)$ by (2) of Theorem 3.7.

We have the following cone theorem for $\overline{\text{NM}}(X)$ and the contraction theorem for *co-extremal rays* ([1],[15]).

Theorem 4.1 (Cone Theorem for $\overline{\text{NM}}(X)$). [1, Theorem 1.1], [15, Theorem 1.3] *Let (X, B) be a dlt pair. There are countably many $(K + B)$ -negative movable curves*

$\{C_i\}_{i \in I}$ such that

$$\overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\text{NM}}(X) = \overline{\text{NE}}(X)_{K+B \geq 0} + \overline{\sum_{i \in I} \mathbb{R}_{\geq 0} \cdot [C_i]}.$$

The rays $\mathbb{R}_{\geq 0} \cdot [C_i]$ can accumulate only along the hyperplanes supporting both $\overline{\text{NM}}(X)$ and $\overline{\text{NE}}(X)_{K+B \geq 0}$.

Remark 4.2. As explained in [1, Example 1.4] and [15, Example 4.9], the genuine form of the cone theorem does not hold for $\overline{\text{NM}}(X)$ in general, that is, we cannot replace $\overline{\text{NE}}(X)$ in Theorem 4.1 by $\overline{\text{NM}}(X)$.

Theorem 4.3 (Contraction theorem for co-extremal faces). [15, Theorem 1.4] *Let (X, B) be a dlt pair. Suppose that F' is a co-extremal face of $\overline{\text{NM}}(X)$ for (X, B) and D be a co-bounding divisor of F' . Then there exists a birational morphism $\varphi : W \rightarrow X$ and a contraction $h : W \rightarrow Z$ such that*

- (1) *Every movable curve C on W with $[\varphi_* C] \in F'$ is contracted by h .*
- (2) *For a general pair of points in a general fiber of h , there is a movable curve C through the two points with $[\varphi_*(C)] \in F'$.*

These properties determine the pair (W, h) , up to a birational equivalence. In fact, the map we construct satisfies a stronger property:

- (3) *There is an open set $U \subseteq W$ such that the complement of U has codimension 2 in a general fiber of h and a complete curve C in U is contracted by h if and only if $[\varphi_* C] \in F'$.*

Remark 4.4. If $K + B \in \partial \overline{\text{Eff}}(X)$, then $\overline{\text{NM}}(X) \subseteq \overline{\text{NE}}(X)_{K+B \geq 0}$ by Theorem 3.7. Thus there are no co-extremal faces for the pair (X, B) . However, there exists an extremal face F' of $\overline{\text{NM}}(X)$ in $\overline{\text{NE}}(X)_{K+B=0}$. If B is big, then $K + B - \varepsilon B$ is not pseudoeffective for any $\varepsilon > 0$ because F' is $(K + (1 - \varepsilon)B)$ -negative. Thus there exists a co-extremal ray of $\overline{\text{NM}}(X)$ for the pair $(X, (1 - \varepsilon)B)$ and since $(X, (1 - \varepsilon)B)$ is klt for small $\varepsilon > 0$, the above theorems can be applied to this pair. In particular, some extremal rays of F' are contractible on some birational model of X .

Theorem 1.2 and Theorem 1.3 can be considered as another analogues of the original Cone Theorem and Contraction Theorem for $\overline{\text{NE}}(X)$ (e.g. [12, Theorem 3.7]). We closely follow the paper [15] in the proofs below. We start with a lemma.

Lemma 4.5. *Let (X, B) be a \mathbb{Q} -factorial klt pair. Let $K + B \notin \overline{\text{Mob}}(X)$ and F be a mov^1 -co-extremal face of $\overline{\text{bNM}}^1(X)$ for (X, B) . If D is a mov^1 -co-bounding divisor of F , then there exists an ample divisor H such that $K + B + H$ is ample and $\alpha D \equiv K + B + cH$ for some $\alpha > 0$ and $0 < c < 1$.*

Proof. Let G be the 2-dimensional closed convex cone in $N^1(X)$ spanned by D and $-(K + B)$. It is enough to prove that the $\emptyset \neq G \cap \text{Amp}(X)$. Indeed, a sufficiently large ample divisor $H \in G \cap \text{Amp}(X)$ satisfies the required conditions.

Suppose that $G \cap \text{Amp}(X) = \emptyset$. Then there exists a curve class L which separates the two cones: $L \cdot D' < 0$ for all $D' \in G \setminus \{0\}$ and $L \cdot D'' > 0$ for all $D'' \in \text{Amp}(X)$. By the second inequality, $L \in \overline{\text{NE}}(X)$. The first inequality with $D' = -(K+B)$ gives $L \in \overline{\text{NE}}(X)_{K+B>0}$. However the first inequality with $D' = D$ also gives $L \cdot D < 0$, contradicting the fact that D is positive on $\overline{\text{NE}}(X)_{K+B>0}$. \square

Conversely, it is easy to see that the divisors $D \in \partial \overline{\text{Mob}}(X)$ of the form $D \equiv K + B + H$ for an ample divisor H are mov^1 -co-bounding divisors of some mov^1 -co-extremal face.

Proposition 4.6. *Let (X, B) be a \mathbb{Q} -factorial klt pair. Consider the cone*

$$V = \overline{\text{NE}}(X)_{K+B+H \geq 0} + \overline{\text{bNM}}^1(X)$$

for some ample divisor H such that $(X, B + H)$ is klt. Then there exists a finite set $\{C_i\}$ of b-mov^1 -curves of X such that for any mov^1 -co-bounding divisor D for some mov^1 -co-extremal face of the cone $\overline{\text{bNM}}^1(X)$ for $(X, B + H)$, $[C_i] \cdot [D] = 0$ for some C_i .

Proof. We may assume that $K + B + H \notin \overline{\text{Mob}}(X)$. Otherwise, $K + B + H$ is nonnegative on $\overline{\text{bNM}}^1(X)$ by Theorem 1.1 and there would be no mov^1 -co-extremal rays. Let D be a mov^1 -co-bounding divisor as in the statement. Then by Lemma 4.5, there exists an ample divisor A such that $K + B + H + A$ is ample and $\alpha D \equiv K + B + H + cA$ for $\alpha > 0$ and $0 < c < 1$. By Lemma 2.3, we may assume that the pair $(X, B + H + A)$ is klt. By Theorem 2.5, there exist a log terminal model $\varphi : X \dashrightarrow X'$ of $(X, B + H + cA)$, which is an isomorphism in codimension 1 since $K + B + H + cA \in \partial \overline{\text{Mob}}(X)$.

There also exists a contraction $\psi : X' \rightarrow Y$ which is either a birational morphism contracting a divisor or has a Mori fiber space structure where $K_{X'} + B_{X'} + H_{X'} + cA_{X'}$ vanishes on every curve contracted by ψ . If $K + B + H + cA$ is big, then there exists a divisorial component $E \subseteq \mathbf{B}_+(K_{X'} + B_{X'} + H_{X'} + cA_{X'})$ by Lemma 3.2 and Proposition 3.3. By [5, Proposition 1.5], the divisor E is ψ -exceptional and there exists a mov^1 -curve C' on X' contracted by ψ . If $K + B + H + cA \in \partial \overline{\text{Eff}}(X)$, then ψ has a Mori fiber space structure and a movable curve C' is contracted by ψ . Thus, in either case, we obtain a b-mov^1 -curve C' of X such that $[C'] \cdot [D] = 0$. By the finiteness (Theorem 2.4), as we vary D in \mathcal{E}_H such that $D \in \partial \overline{\text{Mob}}(X)$, we obtain only finitely many maps $\psi \circ \varphi$ and consequently finitely many b-mov^1 -curves. \square

Proof of Theorem 1.2. We may assume that $K + B \notin \overline{\text{Mob}}(X)$. Otherwise, $K + B$ is nonnegative on $\overline{\text{bNM}}^1(X)$ by Theorem 1.1 and there would be no mov^1 -co-extremal rays. We may assume that $(X, B + H)$ is klt by Lemma 2.3. Let $\{\varepsilon_j\}$ be a sequence of strictly decreasing positive numbers converging to 0. Let $\{C_{ji}\}_{i \in I_j}$ be the finite set of all b-mov^1 -curves for $(X, B + \varepsilon_j H)$ obtained as in Proposition 4.6 using Theorem

2.5 . Then clearly,

$$\overline{\text{NE}}(X)_{K+B+\varepsilon_j H \geq 0} + \overline{\text{bNM}}^1(X) \supseteq \overline{\text{NE}}(X)_{K+B+\varepsilon_j H \geq 0} + \overline{\sum_{i \in I_j} \mathbb{R}_{\geq 0} \cdot [C_{ji}]}.$$

Suppose that the strict inclusion \supsetneq holds. Then there exists a mov^1 -co-extremal ray R for $(X, B + \varepsilon_j H)$ such that $R \setminus \{0\}$ is disjoint from $\overline{\text{NE}}(X)_{K+B+\varepsilon_j H \geq 0} + \overline{\sum_{i \in I_j} \mathbb{R}_{\geq 0} \cdot [C_{ji}]}$. If D is a mov^1 -co-bounding divisor of R , then by Lemma 4.5, there exists an ample divisor A such that $K + B + \varepsilon_j H + A$ is ample and $\alpha D \equiv K + B + \varepsilon_j H + cA$ for $\alpha > 0$ and $0 < c < 1$. Since we may assume that $(X, B + \varepsilon_j H + cA)$ is klt, by applying Theorem 2.5 on $(X, B + \varepsilon_j H + cA)$, we obtain a b-mov^1 -curve C of X (as in the proof of Proposition 4.6) such that $R = \mathbb{R}_{\geq 0} \cdot [C]$. Since $R \notin \{\mathbb{R}_{\geq 0} \cdot [C_{ji}]\}$ and R is also a mov^1 -co-extremal ray for $(X, B + \varepsilon_j H)$, it is a contradiction. So the second equality holds.

Suppose that the set $\cup_{j \in \mathbb{N}} I_j$ is infinite. Since $I_j \subseteq I_{j+1}$ for all j , we may assume that $I_j \subsetneq I_{j+1}$ for all j . By taking the limit $j \rightarrow \infty$, we obtain the second equality of the cones and the last statement. \square

Proof of Theorem 1.3. We may assume that $K + B \notin \overline{\text{Mob}}(X)$. Otherwise, $K + B$ is nonnegative on $\overline{\text{bNM}}^1(X)$ by Theorem 1.1 and there would be no mov^1 -co-extremal rays. For a fixed mov^1 -co-extremal ray R , by Lemma 4.5, there exists an ample divisor H such that $K + B + H$ is ample and $D = K + B + cH$ ($0 < c < 1$) is a mov^1 -co-bounding divisor for R . We may assume that $(X, B + H)$ is klt by Lemma 2.3. By Theorem 2.5, there exists a log terminal model $\varphi : X \dashrightarrow X'$ of $(X, B + cH)$ and since $D \equiv K + B + cH \in \partial \overline{\text{Mob}}(X)$, the birational map $\varphi : X \dashrightarrow X'$ is an isomorphism in codimension 1.

If $D \in \partial \overline{\text{Eff}}(X)$, then the ray R is a co-extremal ray of $\overline{\text{NM}}(X)$ for (X, B) . By [4], there exists a Mori fiber space structure $X' \rightarrow Y$ and the statements follow from Theorem 4.3. Assume that $D \in \text{Int } \overline{\text{Eff}}(X)$. Then as in the proof of Proposition 4.6, the ray R is spanned by a mov^1 -curve C' on X' (which is not movable and is a b-mov^1 -curve of X) and its associated contraction $\psi : X' \rightarrow Y$ is divisorial. Now by the uniqueness of the (lc) model Y (Theorem 2.5) for the mov^1 -co-bounding divisor $D = K + B + cH$ of R , we obtain the statements (1) and (2). \square

Remark 4.7. As illustrated in the Figure 1, there may be an extremal ray R of $\overline{\text{bNM}}^1(X)$ which is not mov^1 -co-extremal, but co-extremal. This ray does not appear in the expression $\overline{\text{NE}}_{K+B \geq 0}(X) + \overline{\sum_{i \in I} \mathbb{R}_{\geq 0} \cdot [C_i]}$ in Theorem 1.2. However, the statements of Theorem 1.3 also hold for this ray by Theorem 4.3.

In the statements of Theorem 1.3, if $K + B$ is not big, then $\psi : X' \rightarrow Y$ is a Mori fibration and this is a resulting model of the given pair (X, B) . Note that if $K + B \in \partial \overline{\text{Eff}}(X)$, then $K_{X'} + B_{X'}$ is ψ -trivial and Y is the lc Iitaka model of (X, B) .

If $K + B$ is big and $K + B \in \overline{\partial \text{Mob}}(X)$, then $(X', B_{X'})$ is a resulting model which is a log terminal model of (X, B) and the contraction $\psi : X' \rightarrow Y$ is the lc contraction to the lc model $Y = X_{\text{lcm}}$ of (X, B) . For all other cases, namely, when $K + B$ is big but not in $\overline{\text{Mob}}(X)$, the divisorial contraction ψ in Theorem 1.3 is only one of the intermediate modifications of the LMMP.

Remark 4.8. If $K + B \in \overline{\text{Mob}}(X)$, then the cone $\overline{\text{bNM}}^1(X)$ does not have any mov^1 -co-extremal faces. However, if $K + B \in \partial \overline{\text{Mob}}(X)$, then $\overline{\text{bNM}}^1(X)$ has extremal faces in $\overline{\text{NE}}(X)_{K+B=0}$. If $K + B$ is big or $B \in \text{Int } \overline{\text{Mob}}(X)$, then some of such faces F are mov^1 -co-extremal for some klt pair and Theorem 1.3 holds for these rays too. Indeed, suppose $\overline{\text{bNM}}^1(X) \subseteq \overline{\text{NE}}(X)_{K+B \geq 0}$ and let F be an extremal face of $\overline{\text{bNM}}^1(X)$ in $\overline{\text{NE}}(X)_{K+B=0}$. If $K + B$ is big, then $K + B \equiv H + E$ for some ample H and effective E . For a small $\varepsilon > 0$, $K + B + \varepsilon E$ is big and $(X, B + \varepsilon E)$ is still klt. However, $K + B + \varepsilon E \notin \overline{\text{Mob}}(X)$ since we can easily check that F is $(K + B + \varepsilon E)$ -negative and $\overline{\text{NE}}(X)_{K+B+\varepsilon E=0}$ does not intersect with the supporting plane $\{[C] \in N_1(X) \mid C \cdot (K + B) = 0\}$. Therefore, F is a mov^1 -co-extremal face of $\overline{\text{bNM}}^1(X)$ for the pair $(X, B + \varepsilon E)$ and $K + B$ is a mov^1 -co-bounding divisor for F . Since the extremal rays of F are mov^1 -co-extremal rays, Theorem 1.2 and Theorem 1.3 can be applied to this case. The similar argument works for the case when $B \in \text{Int } \overline{\text{Mob}}(X)$ (cf. Remark 4.4).

Question 4.9. In [3], Batyrev conjectured that the co-extremal rays in Theorem 4.1 do not accumulate away from $\overline{\text{NE}}(X)_{K+B=0}$. Similarly, we can ask whether the mov^1 -co-extremal rays in Theorem 1.2 can accumulate away from $\overline{\text{NE}}(X)_{K+B=0}$. For the results related to the conjecture of Batyrev or the cone $\overline{\text{NM}}(X)$, see [1],[2],[3],[4],[15],[22].

REFERENCES

- [1] C. Araujo, The cone of pseudo-effective divisors of log varieties after Batyrev, *Math. Z.* 264, no. 1, 179-193, (2010)
- [2] S. Barkowski, The cone of moving curves of a smooth Fano three- or fourfold, *manuscripta mathematica*. Vol 131, Numbers 3-4, 305-322, (2010)
- [3] V.V. Batyrev, The cone of effective divisors of threefolds, *Proceedings of the International Conference on Algebra, Part 3* (Novosibirsk, 1989), *Contemp. Math.*, 131, Part 3, AMS, 337-352, (1992)
- [4] C. Birkar, P. Cascini, C. Hacon, J. McKernan, Existence of minimal models for varieties of log general type, *J. Amer. Math. Soc.* 23, no. 2, 405-468, (2010)
- [5] S. Boucksom, A. Broustet, G. Pacienza, Uniruledness of stable base loci of adjoint linear systems with and without Mori Theory, *arXiv:0902.1142v2 [math.AG]*, (2009, Preprint)
- [6] S. Boucksom, J.-P. Demailly, M. Paun, T. Peternell, The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, to appear in *J. Algebraic Geom.*

- [7] L. Ein, R. Lazarsfeld, M. Mustatǎ, M. Nakamaye, Mihnea Popa, Restricted volumes and base loci of linear series, *Amer. J. Math.* 131, no. 3, 607-651, (2009)
- [8] L. Ein, R. Lazarsfeld, M. Mustatǎ, M. Nakamaye, Mihnea Popa, Asymptotic invariants of base loci, *Ann. Inst. Fourier*, 56 no.6, 1701-1734, (2006)
- [9] Y. Hu, S. Keel, Mori dream spaces and GIT, *Michigan Math. J.* 48, 331-348, (2000)
- [10] V.A. Iskovskikh and V.V. Shokurov, Birational models and flips, *Russian Mathematical Surveys*, 60, 27-94, (2005)
- [11] Y. Kawamata; *Remarks on the cone of divisors*, in *Classification of Algebraic Varieties*, EMS Series of Congress Reports. European Mathematical Society, (2011)
- [12] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge tracts in mathematics, vol. 134, Cambridge University Press, Cambridge UK, (1998)
- [13] R. Lazarsfeld, *Positivity in algebraic geometry I*, 48. xviii+385 pp. Springer-Verlag, Berlin, (2004)
- [14] R. Lazarsfeld, *Positivity in algebraic geometry II*, 49. xviii+387 pp. Springer-Verlag, Berlin, (2004)
- [15] B. Lehmann, A cone theorem for nef curves, arXiv:0807.2294v4 [math.AG], (2011 Preprint)
- [16] K. Matsuki, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, (2002)
- [17] M. Nakamaye, Stable base loci of linear series. *Math. Ann.* 318, no. 4, 837-847, (2000)
- [18] S. Payne, Stable base loci, movable curves, and small modifications, for toric varieties, *Math. Z.* 253, no. 2, 421-431, (2006)
- [19] Yu.G. Prokhorov and V.V. Shokurov, Towards the second main theorem on complements, *J. Algebraic Geom.* 18, no. 1, 151-199, (2009)
- [20] V.V. Shokurov, 3-fold log models, *Algebraic Geometry*, 4, J.Math. Sci. 81, Consultants Bureau, New York, 2667-2699, (1996)
- [21] V.V. Shokurov, S. Choi, Geography of log models: theory and application, *Cent. Eur. J. Math.* 9(3), 489-534, (2011)
- [22] Q. Xie, The Nef Curve Cone Theorem Revisited, arXiv:math/0501193v2 [math.AG], (2005, Preprint)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, RIVERSIDE. 900 UNIVERSITY AVE, RIVERSIDE, CA 92521

E-mail address: schoi@math.ucr.edu